# The almost-rigid rotation of viscous fluid between concentric spheres

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(*Received* 30 *April* 1956)

SUMMARY

Two concentric spheres are supposed to rotate about the same axis with almost the same angular velocity, so that the viscous stresses over the surfaces of the spheres induce a flow which may be represented by a small perturbation superimposed upon a rigid body rotation of the fluid as a whole. The governing equations are therefore linearized in the magnitude of the perturbation, and it appears that the validity of this linearization is independent of the Reynolds number of the primary rotation. Attention is then restricted to the case in which the Reynolds number is large, the principal object of the note being to exemplify some of the properties of rotating systems at large Reynolds numbers in terms of a particularly simple mathematical model.

It is found that the cylindrical surface that touches the inner sphere (the axis being the axis of rotation) is a singular surface in which velocity gradients are very large. Everywhere outside this cylinder, the fluid rotates as a rigid body with the same angular velocity as the outer sphere. Inside the cylinder, the velocity distribution in the central (inviscid) core of the motion is shown to be determined by the velocity distribution in the boundary layers over the spheres, and explicit solutions are obtained for all these velocity distributions. The mechanics of the cylindrical shear layer itself is also discussed, though no explicit solution is obtained in this case.

## 1. INTRODUCTION

In the literature relating to rotating systems in fluid dynamics, very little attention has been paid to problems in which the configuration of the boundaries is such that the flow may be represented by a small perturbation superimposed upon a rigid body rotation of the (incompressible) fluid as a whole. Indeed, the only relevant work of which I am aware is that of Squire (1953) and Stewartson (1953). Squire considered the flow due to a rotating disc when the fluid at a great distance from the disc is rotating as a rigid body about the same axis with almost the same angular velocity, and Stewartson considered the similar problem in which two discs rotate about the same axis with almost the same angular velocity. In both these problems, the potential value of the basic approximation was, perhaps, somewhat obscured by the fact that the general form of the exact solutions of the Navier–Stokes equations had previously been found by Batchelor (1951). The peculiar property of the general class of flows under discussion is, in fact, that the governing equations become linear in a manner that is independent of the Reynolds number, and the much greater tractability of linear equations leads to the expectation that substantial analytical progress can be made in such problems. A discussion of many of the interesting properties of rotating systems is undoubtedly prohibited by the linearization, but this is obviously not true of all such properties. In particular, the phenomenon of flow at large Reynolds numbers being governed by linear equations is of real interest, and may be expected to yield useful information concerning the behaviour of boundary layers in rotating systems.

The present note is concerned with the particular problem in which two concentric spheres rotate rapidly about the same axis with slightly different angular velocities. In such a system, it is clear that the viscous stresses over the boundaries must induce a secondary flow, and the principal purpose of the note is to examine the properties of this secondary flow when the Reynolds number is large.

## 2. The governing equations

Let the radii of the inner and outer spheres be $a$ and $\alpha a$, respectively, and let the corresponding angular velocities be $\Omega$ and $\Omega(1+\epsilon)$, where $\epsilon$ is very small. Let $(r', \theta, \phi)$ denote spherical polar coordinates in which the line $\theta = 0$ coincides with the axis of rotation, and let $u'$, $v'$, $w'$, be the corresponding components of velocity (see figure 1).

By symmetry, all dynamical variables must be independent of $\phi$, so that the incompressibility condition may be integrated in terms of a stream function $\psi'$, where

$$u' = \frac{1}{r'^2 \sin\theta}\frac{\partial\psi'}{\partial\theta}, \qquad v' = -\frac{1}{r'\sin\theta}\frac{\partial\psi'}{\partial r'}. \tag{2.1}$$

Then, writing

$$w' = \frac{\chi'}{r'\sin\theta}, \tag{2.2}$$

the Navier–Stokes equations of steady motion become (see Goldstein 1938, ch. 3)

$$\frac{2\chi'}{r'^2\sin^2\theta}\left[\frac{\partial\chi'}{\partial r'}\cos\theta - \frac{1}{r'}\frac{\partial\chi'}{\partial\theta}\sin\theta\right] - \frac{1}{r'^2\sin\theta}\frac{\partial(\psi', D^2\psi')}{\partial(r',\theta)} + \\ + \frac{2D^2\psi'}{r'^2\sin^2\theta}\left[\frac{\partial\psi'}{\partial r'}\cos\theta - \frac{1}{r'}\frac{\partial\psi'}{\partial\theta}\sin\theta\right] = \nu D^4\psi', \tag{2.3}$$

and

$$-\frac{1}{r'^2\sin\theta}\frac{\partial(\psi',\chi')}{\partial(r',\theta)} = \nu D^2\chi', \tag{2.4}$$

where

$$D^2 = \frac{\partial^2}{\partial r'^2} + \frac{\sin\theta}{r'^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{2.5}$$

The boundary conditions for equations (2.3) and (2.4) are :

$$\left.\begin{aligned} r' &= a, & \frac{\partial\psi'}{\partial r'} &= \psi' = 0, & \chi' &= \Omega a^2\sin^2\theta\,; \\ r' &= \alpha a, & \frac{\partial\psi'}{\partial r'} &= \psi' = 0, & \chi' &= \Omega(1+\epsilon)\alpha^2 a^2\sin^2\theta. \end{aligned}\right\} \tag{2.6}$$

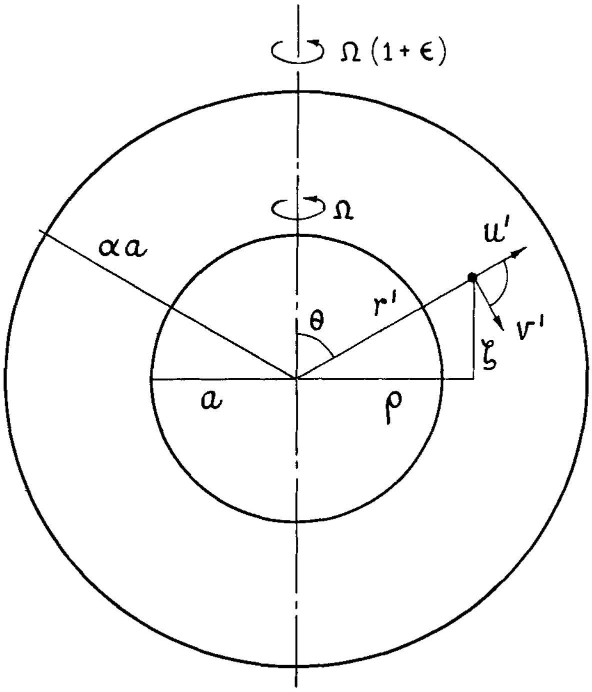


Figure 1. Notation.

When $\epsilon = 0$, the exact solution of the problem is obviously the rigid body rotation represented by

$$\psi' = 0, \qquad \chi' = \Omega r'^2\sin^2\theta.$$

Hence, when $\epsilon$ is non-zero but small, it is reasonable to write

$$\psi' = \epsilon a^3\Omega\psi, \qquad \chi' = \Omega r'^2\sin^2\theta + \epsilon a^2\Omega\chi, \tag{2.7}$$

and retain only terms of the first order in $\epsilon$. Thus, introducing the dimensionless variable

$$r = r'/a, \tag{2.8}$$

the dynamical equations (2·3) and (2·4) reduce to

$$2\left(\frac{\partial\chi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\chi}{\partial\theta}\sin\theta\right) = \frac{1}{R}D^4\psi, \tag{2.9}$$

$$-2\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right) = \frac{1}{R}D^2\chi, \tag{2.10}$$

where $D^2$ is the same operator as (2.5), but with $r'$ replaced by $r$, and $R$ is the Reynold's number given by

$$R = \frac{a^2\Omega}{\nu}. \tag{2.11}$$

The boundary conditions become

$$\left.\begin{aligned} r = 1, \quad & \frac{\partial\psi}{\partial r} = \psi = 0, \quad && \chi = 0, \\ r = \alpha, \quad & \frac{\partial\psi}{\partial r} = \psi = 0, \quad && \chi = \alpha^2 \sin^2\theta. \end{aligned}\right\} \tag{2.12}$$

The equations (2.9) and (2.10) form the basis of the present note.

It is worth noting here that the approximation leading to (2.9) and (2.10) is somewhat analogous to the approximation introduced by Oseen (1910) in problems of uniform streaming past obstacles. For, in both cases, a linear approximation to the inertia terms in the Navier–Stokes equation is made. In Oseen's case, however, it is an immediate (kinematic) consequence of the boundary conditions that the approximate form of the inertia terms is invalid in the neighbourhood of the obstacle, so that the approximation is spatially uniform only if the Reynolds number is very small. In the present case no such troubles seem to arise, since the boundary conditions are kinematically consistent with the linearization over the whole field of flow, and the Reynolds number of the primary motion remains a significant parameter on which there is no immediately obvious restriction. Nevertheless, when the Reynolds number is very large, the possibility that intense velocity gradients in the secondary flow might invalidate the linearization must not be ignored. This last point will be a matter for subsequent examination.

Although the spherical polar coordinates are particularly suited to the geometry of the boundaries, many aspects of the natural dynamical behaviour of the system at large Reynolds numbers are of a much more two-dimensional nature than the boundary conditions suggest. It will therefore be useful to set out here the governing equations in terms of the cylindrical coordinates (see figure 1)

$$\rho = r\sin\theta, \qquad \zeta = r\cos\theta. \tag{2.13}$$

Thus, (2.9) and (2.10) become

$$2\frac{\partial\chi}{\partial\zeta} = \frac{1}{R}D^4\psi, \tag{2.14}$$

and

$$-2\frac{\partial\psi}{\partial\zeta} = \frac{1}{R}D^2\chi, \tag{2.15}$$

where

$$D^2 = \frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial}{\partial\zeta^2}. \tag{2.16}$$

## 3. The boundary layers over the spheres

When the Reynolds number is very large, viscous forces must be negligible over the whole field of flow, except, possibly, in the neighbourhood of certain singular surfaces. The cylindrical polar equations (2.14) and (2.15) then show that, apart from such surfaces, the solutions must take the simple form

$$\psi = \psi(\rho), \qquad \chi = \chi(\rho), \tag{3.1}$$

and since it is clearly impossible to satisfy the viscous boundary conditions (2.12) with such functions, it follows that the singular surfaces must indeed exist in this problem.

Now, it is natural in the first place to suppose that the singular surfaces take the form of conventional boundary layers over the two spherical boundaries. Thus, taking the case of the inner sphere first, the usual boundary layer approximations in the dynamical equations (2.9) and (2.10) lead to the equations

$$2\frac{\partial \chi}{\partial r}\cos\theta = \frac{1}{R}\frac{\partial^4\psi}{\partial r^4}, \qquad -2\frac{\partial \psi}{\partial r}\cos\theta = \frac{1}{R}\frac{\partial^2\chi}{\partial r^2}. \tag{3.2}$$

In view of the large velocity gradients implied by the approximate equations (3.2), it is then proper to enquire whether these equations are consistent with the original linearization in $\epsilon$. Returning, therefore, to the exact dynamical equations (2.3) and (2.4), the neglected inertia terms in the first of these equations are

$$\epsilon^2 a\Omega^2\left[\frac{2\chi}{r^2\sin^2\theta}\left\{\frac{\partial\chi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\chi}{\partial\theta}\sin\theta\right\} - \frac{1}{r^2\sin^2\theta}\frac{\partial(\psi, D^2\psi)}{\partial(r,\theta)} + \right.$$
$$\left. + \frac{2D^2\psi}{r^2\sin^2\theta}\left\{\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right\}\right],$$

which, on the assumption that a boundary layer exists at $r = 1$, reduce to

$$\epsilon^2 a\Omega^2\left[\frac{2\cos\theta}{\sin^2\theta}\chi\frac{\partial\chi}{\partial r} - \frac{1}{\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial^3\psi}{\partial r^3} - \frac{\partial\psi}{\partial r}\frac{\partial^3\psi}{\partial r^2\partial\theta}\right) + \frac{2}{\sin^2\theta}\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial r^2}\right].$$

Thus, since we may assume

$$\psi = O(R^{-1/2}), \qquad \chi = O(1),$$
$$\partial/\partial r = O(R^{1/2}), \qquad \partial/\partial\theta = O(1),$$

the order of magnitude of the neglected terms is

$$\epsilon^2 a\Omega^2 R^{1/2}.$$

The order of magnitude of the inertia term retained in the first of equations (3.2), on the other hand, is

$$\epsilon a\Omega^2\frac{\partial\chi}{\partial r}\cos\theta \quad \text{or} \quad \epsilon a\Omega^2 R^{1/2}.$$

Hence the neglected terms are smaller by a factor $\epsilon$ than those retained, and the presence of large velocity gradients does not affect the basic approximation. The argument may not, of course, apply near particular values of $\theta$, notably near the equator. Similar remarks apply to the other dynamical equation.

The property of the equations (3.2) that is chiefly responsible for the success of this investigation is that they are both *ordinary* differential equations. This remarkable circumstance possibly embodies a point of principle which might have been anticipated on general grounds. It is, of course, a consequence of the linearization that the entire secondary flow must be reversible. In particular, this reversibility must be true of the flow in the boundary layers. Now, the boundary layer flow in an axial plane presumably involves a general mass flux from the poles to the equator, or *vice versa*, depending on whether $\epsilon$ is negative or positive (for the inner sphere). Hence, if one regards this flow as being determined by conditions just outside the boundary layer, together with an initial condition represented by the fact that the flow is of some generalized 'stagnation' type at the poles (say), then the corresponding solution at the equator must be such that, when used as an initial condition for the reversed flow, it yields an identical flow pattern. This would appear to impose a severe integral condition on the flow in the boundary layers, and by far the most attractive method of satisfying this condition is to suppose that the velocity profile at each value of $\theta$ is determined independently. That such is the case is shown by (3.2), and the explanation given seems at least a possible one.

One integration of (3.2) gives

$$\left.\begin{aligned} 2\cos\theta(\chi-\chi_0) &= \frac{1}{R}\frac{\partial^3\psi}{\partial r^3}, \\ -2\cos\theta(\psi-\psi_0) &= \frac{1}{R}\frac{\partial\chi}{\partial r}, \end{aligned}\right\} \tag{3.3}$$

where $\psi_0$ and $\chi_0$ are functions of $\theta$ only, which are determined by conditions at the outer edge of the boundary layer. From (3.3) it follows that

$$\frac{\partial^4}{\partial r^4}(\psi-\psi_0) = -4R^2\cos^2\theta(\psi-\psi_0), \tag{3.4}$$

so that a suitable measure of distance from the boundary is

$$\eta = (r-1)(R\cos\theta)^{1/2} \quad (0 \leqslant \theta < \tfrac{1}{2}\pi). \tag{3.5}$$

The four independent solutions of (3.4) are then of the form

$$\exp\{(\pm 1 \pm i)\eta\},$$

and the bounded solution for $\psi$ that has a double zero at $\eta = 0$ is

$$\psi = \psi_0\{1-e^{-\eta}(\cos\eta+\sin\eta)\}. \tag{3.6}$$

The corresponding solution for $\psi$ follows immediately from the first of equations (3.3), and is

$$\chi = \chi_0 - 2(R\cos\theta)^{1/2}\psi_0\, e^{-\eta}\cos\eta \tag{3.7}$$

Then, since $\chi$ must also vanish at $\eta = 0$, we must have

$$\chi_0 = 2(R\cos\theta)^{1/2}\psi_0. \tag{3.8}$$

The formulae (3.5) to (3.8), which represent the explicit solution for the flow in a three-dimensional boundary layer, were first obtained by Ekman (1902). Actually, Ekman considered the flow near a rotating disc when the relative flow at a great distance from the disc is a uniform stream, but, to the order of the boundary layer approximation, this flow is clearly identical with the flow near the spherical boundaries of the present problem. Since the boundary layer is very thin, each element of the surface of the sphere may be regarded as a large disc rotating with the normal component $\Omega \cos \theta$ of the angular velocity of the sphere, and the relative velocity outside the boundary layer is locally a uniform stream.

The most striking feature of the solution is the oscillatory approach to conditions in the 'inviscid' core of the motion (the 'Ekman spiral'). The same feature appeared, of course, in Squire's and Stewartson's solutions for the flow near a rotating disc. Another point of interest is the explicit formula (3.5) for the thickening of the boundary layer towards the equator. Very close to the equator this thickening becomes very rapid, and when $\frac{1}{2}\pi - \theta = O(R^{-1})$ the boundary layer approximation clearly breaks down altogether.

It should also be noted at this stage that the relation (3.8), which is demanded by the viscous mechanics of the boundary layer on the inner sphere, implies a relation between the functions $\psi$ and $\chi$ throughout the whole of that part of the core of the motion which lies within the cylinder $\rho = 1$ (i.e. the cylinder that touches the inner sphere). The reason for this is the form (3.1) that the solution must take in the absence of viscous forces. Actually, it is not strictly correct to apply the first of the results (3.1) without further explanation because (3.8) ensures that $\psi = O(R^{-1/2})$ which vanishes in the limit of infinite Reynolds number. The strictly 'inviscid' solution is therefore

$$\psi = 0, \qquad \chi = (\rho).$$

But, of course, when $\psi = O(R^{-1/2})$ and $\chi = O(1)$, it is still true that the viscous terms in the dynamical equation (2.15) are negligible in the core of the motion, so that the first of the results (3.1) is still valid. In this restricted sense, the motion in the core is 'inviscid', and we may write

$$\psi = \psi_0(\rho), \qquad \chi = \chi_0(\rho). \tag{3.9}$$

Since $\rho = \sin\theta$ on the inner sphere, it then follows that $\psi_0(\sin\theta)$ and $\chi_0(\sin\theta)$ are the functions of $\theta$ only that appear in (3.3), so that the condition (3.8) becomes

$$\chi_0(\rho) = 2R^{1/2}(1-\rho^2)^{1/4}\psi_0(\rho) \quad (\rho < 1). \tag{3.10}$$

The analysis of the motion in the boundary layer on the outer sphere is substantially the same. Here, the integration of (3.2) gives

$$\left.\begin{aligned} 2\cos\theta\{\chi - \chi_0(\alpha\sin\theta)\} &= \frac{1}{R}\frac{\partial^3\psi}{\partial r^3}, \\ -2\cos\theta\{\psi - \psi_0(\alpha\sin\theta)\} &= \frac{1}{R}\frac{\partial\chi}{\partial r}, \end{aligned}\right\} \tag{3.11}$$

where $\psi_0$ and $\chi_0$ have the same meanings as in (3.9), since $\alpha\sin\theta$ is the value of $\rho$ on the outer sphere. Then, introducing the variable

$$\eta' = (\alpha - r)(R\cos\theta)^{1/2}, \tag{3.12}$$

the relevant solution for $\psi$ is

$$\psi = \psi_0(\alpha\sin\theta)\{1 - e^{-\eta'}(\cos\eta' + \sin\eta')\}, \tag{3.13}$$

and the corresponding solution for $\chi$ is

$$\chi = \chi_0(\alpha \sin\theta) + 2(\mathrm{R}\cos\theta)^{1/2}\psi_0(\alpha\sin\theta)e^{-\eta'}\cos\eta'. \tag{3.14}$$

Moreover, the boundary conditions (2.12) require that $\chi = \alpha^2\sin^2\theta$ over the outer sphere $\eta' = 0$, so that (3.14) gives

$$\alpha^2\sin^2\theta - \chi_0(\alpha\sin\theta) = 2(R\cos\theta)^{1/2}\psi_0(\alpha\sin\theta). \tag{3.15}$$

Now, the result (3.15) provides a relation between the functions $\psi_0$ and $\chi_0$ which must be satisfied throughout the entire core of the motion. Thus,

$$\rho^2 - \chi_0(\rho) = 2R^{1/2}\left(1-\frac{\rho^2}{\alpha^2}\right)^{1/4}\psi_0(\rho), \tag{3.16}$$

without restriction on the value of $\rho$. In the region $\rho < 1$, therefore, the two relations (3.10) and (3.16) uniquely determine the functions $\psi_0$ and $\chi_0$, and the relevant formulae are readily found to be

$$\psi_0(\rho) = \frac{\rho^2}{2R^{1/2}}\left\{\left(1-\frac{\rho^2}{\alpha^2}\right)^{1/4} + (1-\rho^2)^{1/4}\right\}^{-1}, \tag{3.17}$$

and

$$\chi_0(\rho) = \rho^2(1-\rho^2)^{1/4}\left\{\left(1-\frac{\rho^2}{\alpha^2}\right)^{1/4} + (1-\rho^2)^{1/4}\right\}^{-1}. \tag{3.18}$$

The result (3.18) is particularly interesting inasmuch as it is an explicit example of a velocity distribution of order $R^0$ being determined by viscous mechanics in a case where $R$ is large. The manner in which this takes place in the present problem is, of course, very simple. In the core of the motion, the streamlines in an axial plane are parallel to the axis of rotation, so that fluid leaving the boundary layer on one sphere round the circle $\rho = \rho_0$ must enter the boundary layer on the other sphere round a circle of the same radius. This is one essential connection between the boundary layers on the two spheres. The other essential connection arises from the fact that the azimuthal velocity in the core is also a function only of the radial distance from the axis of rotation, and must be such that the associated centrifugal and coriolis forces are just sufficient to require the same mass flux from *both* boundary layers at any particular value of $\rho$.

It follows from (3.18) that the cylindrical surface $\rho = 1$ must be another singular surface of the motion. It is true that the formula (3.18) is not valid for values of $\rho$ close to unity, but this cannot affect the deduction that velocity gradients become very large at such values of $\rho$. Thus, it seems likely that viscosity must again be taken into account near this surface.

The problem of finding the distribution of $\psi_0$ and $\chi_0$ for values of $\rho$ greater than unity is slightly different, because the result (3.10) is no longer relevant. It may be replaced, however, by the symmetry condition that $\psi = 0$ over the equatorial plane $\zeta = 0$. Hence, either the equatorial plane is also a singular surface of the motion (so that $\psi$ at $\zeta = 0$ is not equal to $\psi_0(\rho)$), or else

$$\psi_0(\rho) \equiv 0 \quad \text{for all } \rho > 1. \tag{3.19}$$

As regards the former possibility, the appropriate boundary layer approximations in the cylindrical polar equations (2.14) and (2.15) yield the equations

$$2R\frac{\partial\chi}{\partial\zeta} = \frac{\partial^4\psi}{\partial\zeta^4}, \qquad -2R\frac{\partial\chi}{\partial\zeta} = \frac{\partial^2\chi}{\partial\zeta^2},$$

so that

$$\frac{\partial^4}{\partial\zeta^4}\{\psi-\psi_0(\rho)\} = -4R^2\{\psi-\psi_0(\rho)\}.$$

But there are clearly no solutions of this last equation that remain bounded for all values of $\zeta$ (both positive and negative), so that (3.19) appears to be the relevant deduction. It follows immediately from (3.16) that

$$\chi_0(\rho) \equiv \rho^2 \quad \text{for all } \rho > 1, \tag{3.20}$$

which shows that the whole of the fluid outside the cylinder containing the inner sphere rotates as a rigid body with the same angular velocity as the outer sphere. There is, of course, no boundary layer over the portion of the outer sphere that bounds this rigid body rotation.

There now remains the problem of finding the motion in the free shear layer near the surface $\rho = 1$. This problem is discussed in the following section.

## 4. The cylindrical shear layer

In the immediate neighbourhood of the surface $\rho = 1$, velocity gradients with respect to $\rho$ must surely be very much greater than those with respect to $\zeta$, so that it is again permissible to make a boundary layer type of approximation in the governing equations. Thus, the equations (2.14) and (2.15) may be written, approximately,

$$2R\frac{\partial\chi}{\partial\zeta} = \frac{\partial^4\psi}{\partial\rho^4}, \qquad -2R\frac{\partial\psi}{\partial\zeta} = \frac{\partial^2\chi}{\partial\rho^2}. \tag{4.1}$$

However, the approximate equations (4.1), unlike those governing the motion in the boundary layers over the spheres, are not always consistent with the linearization in $\epsilon$. For, if the exact equations of motion are written in cylindrical polar coordinates and a shear layer is supposed to exist at $\rho = 1$, it is a simple matter to show that the neglected terms on the left hand sides of the first and second of equations (4.1) are

$$\epsilon R\left(2\chi\frac{\partial\chi}{\partial\zeta} + \frac{\partial\psi}{\partial\rho}\frac{\partial^3\psi}{\partial\rho^2\partial\zeta} - \frac{\partial\psi}{\partial\zeta}\frac{\partial^3\psi}{\partial\rho^3}\right)$$

and

$$\epsilon R\left(\frac{\partial\psi}{\partial\rho}\frac{\partial\chi}{\partial\zeta} - \frac{\partial\psi}{\partial\zeta}\frac{\partial\chi}{\partial\rho}\right),$$

respectively. Then, if $\delta$ is the thickness of the layer, and if $\chi$ is assumed to be $O(1)$ throughout the layer ($\chi$ is actually equal to the azimuthal velocity when $\rho = 1$), the orders of magnitude of these terms are

$$\epsilon R\left(1+\frac{\psi^2}{\delta^3}\right) \quad \text{and} \quad \frac{\epsilon R\psi}{\delta},$$

respectively. Hence, comparing the second of these estimates with the second of equations (4.1), the validity of the linearization is certainly dependent upon the condition that $\epsilon/\delta$ should be small. Thus, if $\delta \to 0$ as $R \to \infty$, the equations (4.1) cannot refer to the limit $R \to \infty$ for fixed small $\epsilon$. However, if, as seems likely, the orders of magnitude of $\psi$, $\chi$ and $\delta$, are determined only by $R$ when $\epsilon$ is small, the equations should apply to the limit* $\epsilon \to 0$ for fixed large $R$. In this sense, the linearized equations (4.1) will be assumed to be valid.

Now, it appears that the boundary layer type of approximation leading to (4.1) is the only feature that the shear layer has in common with conventional boundary layers. It does not seem to be possible, for instance, to have a shear layer of this kind in which dynamical variables are of the same order of magnitude as in ordinary boundary layers. For, if that were the case, we should have

$$\psi = O(R^{-1/2}), \qquad \chi = O(1),$$

together with the information that the thickness of the layer is $O(R^{-1/2})$. But, with these orders of magnitude, the first of equations (4.1) reduces to

$$\frac{\partial^4 \psi}{\partial \rho^4} = 0,$$

and, quite apart from the dynamical unlikelihood of a layer with negligible acceleration in an axial plane, this equation does not possess solutions that are bounded as $(\rho - 1)R^{1/2} \to \pm \infty$.

Again, it might be thought that the orders of magnitude must be such that viscous and inertia forces are comparable in both equations of motion. This would at least correspond in principle, if not in detail, to the mechanics of an ordinary three-dimensional boundary layer. Under these circumstances, elimination of $\psi$ from the equations (4.1) gives

$$-4R^2 \frac{\partial^2 \chi}{\partial \zeta^2} = \frac{\partial^6 \chi}{\partial \rho^6},$$

from which it follows immediately that the thickness of the layer must be $O(R^{-1/3})$, and hence, from (4.1), that

$$\psi/\chi = O(R^{-1/3}). \qquad (4.2)$$

Now, it must surely be the case that $\chi = O(1)$ throughout the layer, so that (4.2) gives $\psi = O(R^{-1/3})$, which implies a general mass flux by the velocity components in an axial plane of the same order of magnitude. But an important *raison d'etre* of the shear layer is to carry fluid from the boundary layer on one sphere to that on the other, and *this* mass flux is only $O(R^{-1/2})$. The conclusion would have to be, therefore, that, to the order of the boundary layer approximation, the total mass flux parallel to the axis of rotation was zero, and this would imply that there was a substantial 'return flow' in the shear layer. I have not been able to show that such a situation is impossible, but the assumption of comparable viscous and inertia forces does seem to

* A somewhat easier limiting process to realize in any practical arrangement.

be a most unnatural method of attempting to satisfy the conditions of the problem.

The only acceptable alternatives appear to be those in which the orders of magnitude of $\psi$ and $\chi$ in the layer are those suggested by the boundary conditions, namely

$$\psi = O(R^{-1/2}), \qquad \chi = O(1), \tag{4.3}$$

and the thickness of the layer is sufficiently large to ensure that viscous forces are comparable with inertia forces in either only one, or possibly neither, of the dynamical equations (4.1). Taking first the case in which the mechanics is not entirely inviscid, it is a simple matter to see that the equation in which viscous forces are appreciable must be the second of equations (4.1); from which it follows that the thickness of the layer is $O(R^{-1/4})$. The governing equations may therefore be written

$$\frac{\partial \chi}{\partial \zeta} = 0, \qquad -2\frac{\partial \psi'}{\partial \zeta} = \frac{\partial^2 \chi}{\partial \rho'^2}, \tag{4.4}$$

where
$$\rho' = (\rho - 1)R^{1/4} \tag{4.5}$$

and
$$\psi' = \psi R^{1/2}. \tag{4.6}$$

If this interpretation of the mechanics is correct, it appears that viscous layers in rotating systems can have an entirely different structure from ordinary boundary layers. This structure is a highly anisotropic one in which viscous forces oppose only the rotary motion, so that the motion in planes containing the axis of rotation remains 'inviscid'. In the special problem under discussion, the inviscid nature of the flow in an axial plane provides a neat explanation of how it is that fluid travels from one sphere to the other within a layer which remains thin. The essential point is that the layer is sufficiently thick, and the velocity sufficiently small, to make viscous forces negligible.

The general solution of (4.4) is easily seen to be

$$\left.\begin{aligned} \chi &= F(\rho'), \\ \psi' &= -\tfrac{1}{2}\zeta F''(\rho') + G(\rho'), \end{aligned}\right\} \tag{4.7}$$

where $F$ and $G$ are arbitrary functions of $\rho'$, and $F''$ is the second derivative of $F$. The form of these arbitrary functions is presumably determined by conditions at the points $\zeta = 0$ and $\zeta = (\alpha^2 - 1)^{1/2}$ where the shear layer joins the boundary layer on the spheres. Unfortunately, the solution for the flow in the neighbourhood of these points lies outside the scope of the boundary layer approximations, and would require a much more comprehensive analysis of the motion than I have been able to make. However, it can at least be shown that the structure of the solution (4.7) is consistent with the boundary conditions as $\rho' \to \pm\infty$. As $\rho' \to +\infty$, the motion must tend to the rigid body rotation represented by (3.19) and (3.20), so that we must have

$$F(\infty) = 1, \qquad G(\infty) = 0.$$

As $\rho' \to -\infty$, on the other hand, the boundary conditions are found by writing $\rho = 1$ in the formulae (3.17) and (3.18). In this way we get

$$F(-\infty) = 0, \qquad G(-\infty) = \frac{1}{2}\left(1 - \frac{1}{\alpha^2}\right)^{-1/4}.$$

It may then be noted that $F''(\rho') \to 0$ as $\rho' \to \pm\infty$, so that the dependence of the solution (4.7) on $\zeta$ disappears as the inviscid regions are approached.

The remaining possibility is that the mechanics of the shear layer is entirely inviscid. This clearly corresponds to the case in which the thickness of the layer is of greater order of magnitude than $R^{-1/4}$. In such a case, both the thickness of the layer and the velocity distribution within it (now of the simple form (3.9)) would presumably be determined by conditions near the joins of the shear layer with the boundary layers over the spheres.

It would seem that the only satisfactory way of deciding which, if either, of the preceding two mechanical structures is correct is to determine the asymptotic form, for large Reynolds numbers, of the exact solution of the linearized equations (2.9) and (2.10).

I am indebted to Mr W. W. Wood for several helpful comments on an earlier draft of this paper.

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